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# Exact propagating dromion-like localized wave solutions of generalized (2 + 1)-dimensional Davey–Stewartson equations

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ABSTRACT

The capacity of the double-Exponential (Exp) function method as an alternative approach to obtain the analytic solutions of higher dimensional coupled nonlinear partial differential equations in Engineering mathematics has been revealed. We employ the double-Exp method aided with symbolic computation to construct dromion-like localized solutions for a (2 + 1)-dimensional generalized Davey–Stewartson (2DGDS) system of equations governing the dynamics of weakly nonlinear modulation of a lattice wave packet in a multi-dimensional lattice. The multidromion-like solutions obtained by the present approach are more general than those obtained earlier.



## 1. Introduction

Nonlinear wave equations play a major role in various fields such as plasma physics, fluid mechanics, optical fibers, solid state physics, chemical kinetics, geochemistry, nonlinear optics and so on [1–7]. Much work has been down over the years on the subject of obtaining the analytical solutions to the nonlinear partial differential equations (NPDEs). One of the most exciting advances of nonlinear science and theoretical physics has been a development of methods to look for exact solutions for NPDEs. The advent of symbolic computation also enables to perform some complicated and tedious algebraic coupled with differential calculations. Various powerful methods for obtaining explicit traveling solitary wave solutions such as inverse scattering transform [8], Hirota bilinear form [9], binary Darboux transformation [10], and Bäcklund transformation [11] have already been used to handle these types of NPDEs. In past few decades, many effective methods such as the sine–cosine method [12,13], Jacobi elliptic function method [14], variational iteration method [15], homotopy perturbation method [16], homogeneous balance method [17], pseudo-spectral method [18], modified extended tanh-function method [19–22] and so on with minimal algebra have been established for obtaining solitary wave solutions. In recent years much attention has been paid to coherent structures in multidimensional lattices (see, e.g. Ref. [23]). Since the pioneering work of Boiti et al. [24], the study of soliton-like structures in higher dimensions has attracted much more attention. In particular, for some (2 + 1)-dimensional integrable models such as the Davey–Stewartson (DS) equations [25], Kadomtsev–Petaevkii (KP) equations [26], Nizhnik–Novikov–Veselov (NNV) equation [27], the exponentially localized soliton solutions called dromions are found by using different approaches. In an interesting development Boiti et al. [24] considered the DS equation and they were able to construct solutions that decay exponentially in all directions. Later the name dromion has been proposed by Fokas and Santini [28] for such solutions which is located at

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the intersection of two perpendicular plane waves [29]. The introduction of the concept of dromions has triggered renewed interest in (2+1)-dimensional soliton systems and the question arises in this connection whether there exists exponentially localized structures in (2 + 1)-dimensional nonintegrable systems. Further, the advent of explode decay dromions using arbitrary functions of space and time variables have set in motion the process of unearthing more and more localized entities in (2 + 1)-dimensional nonlinear systems [30]. Conceived by this in mind, in this paper we employ the double-exponential function method (double-EXP) aided with symbolic computation to obtain exact multidromion-like traveling wave solutions of the generalized Davey–Stewartson (GDS) equations in (2+1)-dimensional lattices with cubic and quartic potentials [31]. The outline of the paper is as follows. In the next section we briefly summarize the double-Exp method. In Section 3, we implement the method to the GDS equations governing the dynamics of nonlinear modulation of multidimensional lattice waves that has been originated by the interaction between a long wavelength acoustic mode and a high frequency mode [31] and obtain a series of exponentially decaying localized dromion-like structures. In Section 4, we conclude our results.

## 2. Review of the double Exp-function method

As remarked above, one of the most efficient ways of solving the NPDEs is the algebraic method like double-EXP method embedded with symbolic computation. The exponential method was first proposed by Wu and He [32]. Further through vigorous research in the last few years, He et al., proposed a study on the concepts of the developed asymptotic methods including exponential function [33] and Zhou proposed an improved form [34]. This paper proposes a double-Exp function method, an extension of the exponential function method for seeking exact solutions with multiple velocities and multiple frequencies in the higher dimensional lattices. Very recently Fu and Dai [35] proposed the double-Exp method and a brief summary of the method will be useful for further discussions. For a given partial differential equation in the form,

$$P(u, u_t, u_x, u_y, u_{xx}, u_{yy}, u_{xy}, v, v_t, v_x, v_y, v_{xx}, v_{yy}, v_{xy}, \ldots) = 0, \tag{1}$$

involves four steps to determine the solutions of $u(x, y, t)$ and $v(x, y, t)$ explicitly where $P$ is a polynomial in its arguments.

*Step* 1: A special form of ansatz with two velocities and frequencies for a wave assumed can be expressed as,

$$u(x, y, t) = \frac{\sum_{i=-m}^{n} [a_i \exp(i\xi) + b_i \exp(i\eta)]}{\sum_{j=-p}^{q} [c_j \exp(j\xi) + d_j \exp(j\eta)]}, \tag{2}$$

and

$$v(x, y, t) = \frac{\sum_{i'=-m'}^{n'} [e_{i'} \exp(i'\xi) + f_{i'} \exp(i'\eta)]}{\sum_{j'=-p'}^{q'} [g_{j'} \exp(j'\xi) + h_{j'} \exp(j'\eta)]}. \tag{3}$$

Here

$$\xi = k_1 \left[ x + l_1 y - 2 \left\{ (p_1 + p_2)\gamma_{11} + (q_1 + q_2) l_1 \gamma_{22} + \frac{1}{2}[(p_1 + p_2) l_1 + q_1 + q_2]\gamma_{12} \right\} t \right],$$

$$\eta = k_2 \left[ x + l_2 y - 2 \left\{ (p_1 + p_2)\gamma_{11} + (q_1 + q_2) l_2 \gamma_{22} + \frac{1}{2}[(p_1 + p_2) l_2 + q_1 + q_2]\gamma_{12} \right\} t \right],$$

where $m, n, p, q, m', n', p'$ & $q'$ are the unknown positive integers to be determined and $a_i, b_i, c_j, d_j, e_{i'}, f_{i'}, g_{j'}$ and $h_{j'}$ are unknown constants respectively.

*Step* 2: According to homogeneous balance principle, the linear term of highest order is balanced with the highest order nonlinear term inorder to determine the values of $m, n, p, q, m', n', p'$, and $q'$.

*Step* 3: Substituting the values of $m, n, p, q, m', n', p'$, & $q'$ and equating the coefficients of exponential terms to zero, a system of algebraic equations involving the parameters $a_i, b_i, c_j, d_j, e_{i'}, f_{i'}, g_{j'}, h_{j'}, k_1, k_2, l_1, l_2, p_1, p_2, q_1, q_2, s_1$ and $s_2$ are obtained and by solving the system of equations using symbolic computation the constant parameters will be determined.

*Step* 4: The constant values are then substituted in the ansatz and the traveling wave solutions are constructed. To check the correctness of the solution it is necessary to substitute them into the original equations.

## 3. Implementation of double Exp-function method to the generalized (2 + 1)-dimensional GDS equations

### 3.1. The physical model

From a physical point of view, the study of propagation of nonlinear waves in higher dimensional systems is an exciting and important task, because the transverse effects must be incorporated to model real systems of more than one-space dimension. In this context, Huang et al., investigated weakly nonlinear modulation of a lattice wave packet by using the quasidiscrete multiple-scale method to derive the following (2 + 1)-dimensional equations [31],

$$\alpha_{11}u_{xx} + \alpha_{22}u_{yy} = -2\beta_1(|v|^2)_x - 2\beta_2(|v|^2)_y,$$
$$iv_t + \gamma_{11}v_{xx} + \gamma_{22}v_{yy} + \gamma_{12}v_{xy} = v(\beta_1 u_x + \beta_2 u_y) + \chi|v|^2 v, \tag{4}$$

where $x$ and $y$ are the spatial co-ordinates and $t$ is time. The coefficients that appear in Eqs. (4) are given by,

$$\alpha_{11} = \frac{1}{\omega}(J_2 - v_g^2), \qquad \alpha_{22} = \frac{J_2}{\omega},$$
$$\beta_1 = \frac{2J_3}{\omega}[\lambda_1(1-\cos q_1') + \lambda_2(1-\cos q_2')],$$
$$\beta_2 = \frac{2J_3}{\omega}[\lambda_1(1-\cos q_2') - \lambda_2(1-\cos q_1')],$$
$$\gamma_{11} = \frac{1}{2\omega}[-v_g^2 + J_2(\lambda_1^2\cos q_1' + \lambda_2^2\cos q_2')],$$
$$\gamma_{22} = \frac{J_2}{2\omega}[\lambda_2^2\cos q_1' + \lambda_1^2\cos q_2'],$$
$$\gamma_{12} = \frac{J_2}{2\omega}\lambda_1\lambda_2(\cos q_2' - \cos q_1'),$$
$$\chi = \frac{2}{\omega}\{2\alpha J_3[(1-\cos q_1')\sin q_1' + (1-\cos q_2')\sin q_2'] + 3J_4[(1-\cos q_1')^2 + (1-\cos q_2')^2]\},$$
$$\alpha = \frac{4J_3[\sin q_1'(1-\cos q_1') + \sin q_2'(1-\cos q_2')]}{4[\omega(q)]^2 - [\omega(2q)]^2},$$
$$\lambda_j = \frac{v_{j''}}{v_g} = \frac{\sin q_{j''}'}{\sqrt{\sin^2 q_1' + \sin^2 q_2'}},$$
$$\text{and} \quad v_g = \frac{1}{\omega}\sum_{j''} J_{2j}\sin(q_{j''}')\frac{\mathbf{a}_{\mathbf{j}''}}{a_{j''}},$$

where $q_{j''}'$ is the wave vector and $\omega$ represents the frequency of the respective harmonic holds with $j'' = 1, 2$. The real function $u(x, y, t)$ stands for a mean motion induced by the oscillatory wave packet, which has the complex envelope function $v(x, y, t)$ and $v_g$, $v_{j''}$ represent the group velocity of the carrier wave and $j''$th component of the group respectively. The parameters $J_1, J_2$ and $J_3$ are associated with the harmonic, cubic and quartic nearest neighbor force constants. Eqs. (4) presented here are a more general form of DS equations than those obtained in water waves [25], which are isotropic in nature. They include the dispersion, diffraction, nonlinearity arising from cubic and quartic interatomic potentials and inherent anisotropy of the system. In the case of pure quartic potential the evolution equations for $u$ and $v$ are decoupled and Eqs. (4) in this case reduce to a generalized (2 + 1)-dimensional nonlinear Schrödinger equations. For the specific choices of system parameters $q_1, q_2, \beta_{j''}, \gamma_{i''j''}, \alpha_{i''j''}, i'' = 1, 2$, Eqs. (4) can also be identified with the conventional DSI and DSII equations appearing in the theory of water waves [25]. It is also proved that Eqs. (4) are conditionally integrable and the existence of soliton solutions requires the following conditions;

$$\gamma_{12} = 4\beta_1\beta_2, \qquad \frac{\alpha_{11}}{\gamma_{11} - 2\beta_1^2} = \frac{\alpha_{22}}{\gamma_{22} - 2\beta_2^2}.$$

### *3.2. The localized structures of 2DGDS*

It is of interest to find essentially (2 + 1)-dimensional solutions of these Eqs. (4) other than this integrability condition in a more general case. Therefore we would like to construct the more generalized localized structures other than the integrability condition by implementing the double-Exp method. We recast Eqs. (4) into the following equivalent form

$$\alpha_{11}\left(k_1^2\psi_{\xi\xi} + 2k_1k_2\psi_{\xi\eta} + k_2^2\psi_{\eta\eta}\right) + \alpha_{22}\left(k_1^2l_1^2\psi_{\xi\xi} + 2k_1k_2l_1l_2\psi_{\xi\eta} + k_2^2l_2^2\psi_{\eta\eta}\right)$$
$$+ 2\beta_1\left(2k_1\phi\phi_\xi + 2k_2\phi\phi_\eta\right) + 2\beta_2\left(2k_1l_1\phi\phi_\xi + 2k_2l_2\phi\phi_\eta\right) = 0, \tag{5}$$

and

$$-(s_1+s_2)\phi + \gamma_{11}\left(k_1^2\phi_{\xi\xi} + 2k_1k_2\phi_{\xi\eta} + k_2^2\phi_{\eta\eta}\right) - \gamma_{11}\left(p_1+p_2\right)^2\phi + \gamma_{22}\left(k_1^2l_1^2\phi_{\xi\xi} + 2k_1l_1k_2l_2\phi_{\xi\eta} + k_2^2l_2^2\phi_{\eta\eta}\right)$$
$$- \gamma_{22}\left(q_1+q_2\right)^2\phi + \gamma_{12}\left(k_1^2l_1\phi_{\xi\xi} + k_1k_2l_2\phi_{\xi\eta} + k_2k_1l_1\phi_{\xi\eta} + k_2^2l_2\phi_{\eta\eta}\right) - \gamma_{12}\left(p_1+p_2\right)\left(q_1+q_2\right)\phi$$
$$- \phi\Big[\beta_1\left(k_1\psi_\xi + k_2\psi_\eta\right) + \beta_2\left(k_1l_1\psi_\xi + k_2l_2\psi_\eta\right)\Big] - \chi\phi^3 = 0, \tag{6}$$

by redefining the functions

$$u(x, y, t) = \psi(\xi, \eta), \tag{7}$$

$$v(x, y, t) = \phi(\xi, \eta)\exp[i(\sigma + \rho)], \tag{8}$$

and

$$\begin{aligned}
\xi &= k_1\left[x + l_1 y - 2\left\{(p_1+p_2)\gamma_{11} + (q_1+q_2)l_1\gamma_{22} + \frac{1}{2}[(p_1+p_2)l_1 + q_1 + q_2]\gamma_{12}\right\}t\right],\\
\eta &= k_2\left[x + l_2 y - 2\left\{(p_1+p_2)\gamma_{11} + (q_1+q_2)l_2\gamma_{22} + \frac{1}{2}[(p_1+p_2)l_2 + q_1 + q_2]\gamma_{12}\right\}t\right],\\
\sigma &= p_1 x + q_1 y + s_1 t,\\
\rho &= p_2 x + q_2 y + s_2 t,
\end{aligned} \tag{9}$$

where $k_1, k_2, l_1, l_2, p_1, p_2, q_1, q_2, s_1$ and $s_2$ are constant parameters to be determined later. Note that $\xi$, $\eta$, $\sigma$ and $\rho$ are the traveling wave variables with multiple velocities and frequencies. Then the functions $\psi$ and $\phi$ are assumed to be rational functions of double exponential represented by $\exp(\xi)$ and $\exp(\eta)$. Eventually, we assume the traveling wave ansatz in the line of double-Exp function method for $\psi$ and $\phi$ as,

$$\phi = \frac{\sum_{i=-m}^{n}[a_i\exp(i\xi) + b_i\exp(i\eta)]}{\sum_{j=-p}^{q}[c_j\exp(j\xi) + d_j\exp(j\eta)]}, \tag{10}$$

$$\psi = \frac{\sum_{i'=-m'}^{n'}[e_{i'}\exp(i'\xi) + f_{i'}\exp(i'\eta)]}{\sum_{j'=-p'}^{q'}[g_{j'}\exp(j'\xi) + h_{j'}\exp(j'\eta)]}, \tag{11}$$

where $m, n, p, q, m', n', p', q'$ are unknown positive integers to be determined and $a_i, b_i, c_j, d_j, e_{i'}, f_{i'}, g_{j'}$ and $h_{j'}$ are unknown constants. Now balancing the linear term of higher order with the highest order nonlinear terms $\phi^3$ in Eq. (6) and by manipulation we have

$$\phi^3 = \frac{A_1\exp[-(3m+p)\xi] + \cdots}{A_2\exp[-4p\xi] + \cdots},$$

and

$$\phi_{\xi\xi} = \frac{B_1\exp[-(m+3p)\xi] + \cdots}{B_2\exp[-4p\xi] + \cdots},$$

where $A_i$ and $B_i$ are the coefficients determined. Equating the powers of exponential terms we have $3m+p = m+3p$ which leads to $m = p$. In a similar manner, again on balancing the terms, we have

$$\phi^3 = \frac{\cdots + C_1\exp[(3n+q)\xi]}{\cdots + C_2\exp[4q\xi]},$$

and

$$\phi_{\xi\xi} = \frac{\cdots + D_1\exp[(3q+n)\xi]}{\cdots + D_2\exp[4q\xi]},$$

here also $C_i$ and $D_i$ are determined coefficients. Now equating the powers of exponential terms we have $3n + q = 3q + n$ which leads to $n = q$. Ultimately, for the other case of traveling wave $\psi$, we equate the linear term $\phi_{\xi\xi}$ with the next higher order nonlinear term $\phi\psi_\xi$ in order to determine the unknown positive integers $m'$, $n'$, $p'$ and $q'$ where

$$\phi_{\xi\xi} = \frac{B_1\exp[-(m+3p)\xi] + \cdots}{B_2\exp[-4p\xi] + \cdots},$$

and on some algebraic manipulation the nonlinear term takes the form

$$\phi\psi_\xi = \frac{E_1\exp[-(3p - p' + m' + m)\xi] + \cdots}{E_2\exp[-4p\xi] + \cdots},$$

where $E_i$ and $B_i$ are determined coefficients and equating the powers of exponential we get $m+3p=3p-p'+m'+m$ that leads to $p'=m'$. Similarly,

$$\phi_{\xi\xi}=\frac{\cdots+D_1\exp[(3q+n)\xi]}{\cdots+D_2\exp[4q\xi]},$$

and

$$\phi\psi_{\xi}=\frac{\cdots+F_1\exp[(3q-q'+n'+n)\xi]}{\cdots+F_2\exp[-4q\xi]},$$

where $F_i$ and $D_i$ are determined coefficients and equating the exponential powers we get, $3q+n=3q-q'+n'+n$ which gives $q'=n'$. Also equating the higher order linear term $\psi_{\xi\xi}$ with the highest order nonlinear term $\phi\phi_{\xi}$ in Eq. (5), we have

$$\psi_{\xi\xi}=\frac{G_1\exp[-(3p'+m')\xi]+\cdots}{G_2\exp[-4p\xi]+\cdots},$$

and on some simplification yields

$$\phi\phi_{\xi}=\frac{H_1\exp[-(-2p+2m+4p')\xi]+\cdots}{H_2\exp[-4p\xi]+\cdots},$$

where $G_i$ and $H_i$ are the determined coefficients and equating the exponential powers we get $3p'+m'=-2p+2m+4p'$ which yield $-p'+m'=-2p+2m$ but from the above results we know that $p=m$, therefore we get $p'=m'$. In a similar fashion,

$$\psi_{\xi\xi}=\frac{\cdots+I_1\exp[(3q'+n')\xi]}{\cdots+I_2\exp[4q'\xi]},$$

and

$$\phi\phi_{\xi}=\frac{\cdots+J_1\exp[(2n+4q'-2q)\xi]}{\cdots+J_2\exp[4q'\xi]},$$

here also $I_i$ and $J_i$ are the determined coefficients and by equating the exponential powers we get $3q'+n'=2n+4q'-2q$ which in turn yields $-q'+n'=2n-2q$. Since from the earlier calculation, we know that $q=n$ henceforth it reads $q'=n'$. The values of $p$, $q$, $p'$ and $n'$ can be freely to choose in general. For convenience, we have opted to choose

$$m=p=n=q=m'=p'=n'=q'=1,$$

and thus one can write the form of $\phi$ and $\psi$ as

$$\phi=\frac{a_{-1}\exp(-\xi)+b_1\exp(\xi)+a_0+b_{-1}\exp(-\eta)+b_1\exp(\eta)}{c_{-1}\exp(-\xi)+c_1\exp(\xi)+b_0+d_{-1}\exp(-\eta)+d_1\exp(\eta)},\tag{12}$$

and

$$\psi=\frac{e_{-1}\exp(-\xi)+e_1\exp(\xi)+e_0+f_{-1}\exp(-\eta)+f_1\exp(\eta)}{g_{-1}\exp(-\xi)+g_1\exp(\xi)+g_0+h_{-1}\exp(\eta)+h_1\exp(\eta)}.\tag{13}$$

*Case* (1): $a_{-1}=b_{-1}=c_{-1}=d_{-1}=e_{-1}=f_{-1}=g_{-1}=h_{-1}=0$.

Now we choose the parameters $a_{-1}, b_{-1}, c_{-1}, d_{-1}, e_{-1}, f_{-1}, g_{-1}$ and $h_{-1}$ vanish for this case and inserting Eqs. (12) and (13) in Eqs. (5) and (6), we obtain the following equations in the powers of exponential as,

$$\begin{aligned}
&M_1\exp(2\eta+3\xi)+M_2\exp(\eta+5\xi)+M_3\exp(\xi+4\eta)+M_4\exp(4\xi+\eta)\\
&\quad+M_5\exp(\eta+2\xi)+M_6\exp(\eta+\xi)+M_7\exp(3\eta+\xi)+M_8\exp(4\eta)\\
&\quad+M_9\exp(\eta+3\xi)+M_{10}\exp(2\xi)+M_{11}\exp(4\xi)+M_{12}\exp(3\xi)+M_{13}\exp(2\eta)\\
&\quad+M_{14}\exp(4\xi+2\eta)+M_{15}\exp(5\xi)+M_{16}\exp(\xi+5\eta)+M_{17}\exp(2\xi+3\eta)\\
&\quad+M_{18}\exp(5\eta)+M_{19}\exp(3\eta+3\xi)+M_{20}\exp(2\xi+4\eta)+M_{21}\exp(2\eta+2\xi)\\
&\quad+M_{22}\exp(\xi)+M_{23}\exp(3\eta)+M_{24}\exp(\eta)+M_{25}\exp(\xi+2\eta)=0,\\
&\text{and}\\
&N_1\exp(3\eta+\xi)+N_2\exp(2\eta+3\xi)+N_3\exp(5\xi)+N_4\exp(\xi)+N_5\exp(\xi+4\eta)\\
&\quad+N_6\exp(2\eta+2\xi)+N_7\exp(2\xi+3\eta)+N_8\exp(\eta)+N_9\exp(4\eta)+N_{10}\exp(\eta+3\xi)\\
&\quad+N_{11}\exp(\xi+2\eta)+N_{12}\exp(5\eta)+N_{13}\exp(2\eta)+N_{14}\exp(\eta+\xi)+N_{15}\exp(2\xi)\\
&\quad+N_{16}\exp(3\xi)+N_{17}\exp(\eta+2\xi)+N_{18}\exp(3\eta)+N_{19}\exp(4\xi)+N_{20}=0,
\end{aligned}\tag{14}$$

where $M_i$ and $N_j$ ($i = 1, \ldots, 25$ & $j = 1, \ldots, 20$) are the determined coefficients. Since the expressions for the coefficients are very lengthy, running to sixty pages, we desist from presenting them here. Now we equate the coefficients of exponential terms to zero and we obtain the hierarchy of equations as follows

$$\begin{aligned}
&M_1 = 0,\\
&M_2 = 0,\\
&\vdots\\
&M_{25} = 0,\\
&\text{and}\\
&N_1 = 0,\\
&N_2 = 0,\\
&\vdots\\
&N_{20} = 0.
\end{aligned} \tag{15}$$

The above equations are cumbersome to solve and hence using symbolic computation, we solve the above systems of algebraic equations and obtain a class of series of localized structures of solutions for different choices of parametric values.

*Case* (1.1):

$$\begin{aligned}
&\Bigg\{ h_1 = \frac{g_0 d_1}{b_0}, b_1 = -\frac{a_0 d_1}{b_0}, k_1 = 0, a_1 = \frac{c_1 a_0}{b_0}, e_1 = \frac{e_0 c_1}{b_0}, g_1 = \frac{c_1 g_0}{b_0},\\
&s_2 = -\frac{\begin{bmatrix} \chi a_0^2 + \gamma_{12} p_2 q_1 b_0^2 + \gamma_{12} p_1 q_2 b_0^2 + \gamma_{12} p_1 q_1 b_0^2 + \gamma_{12} p_2 q_2 b_0^2 + s_1 b_0^2 + \gamma_{11} p_1^2 b_0^2 \\ + 2\gamma_{11} p_1 p_2 b_0^2 + \gamma_{11} p_2^2 b_0^2 + 2\gamma_{22} q_1 q_2 b_0^2 + \gamma_{22} q_1^2 b_0^2 + \gamma_{22} q_2^2 b_0^2 \end{bmatrix}}{b_0^2},\\
&f_1 = \frac{d_1 (8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^3 k_2 (\alpha_{22} l_2^2 + \alpha_{11})}, s_1 = s_1, q_1 = q_1,\\
&q_2 = q_2, p_1 = p_1, p_2 = p_2, d_1 = d_1, l_1 = l_1, l_2 = l_2, c_1 = c_1, k_2 = k_2 \Bigg\}.
\end{aligned}$$

Using the above constant parameters, we rewrite Eqs. (7) and (8) as

$$u = \frac{\left(\frac{e_0 c_1}{b_0}\right)\exp(\xi) + e_0 + \left(\frac{d_1(8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^3 k_2 (\alpha_{22} l_2^2 + \alpha_{11})}\right)\exp(\eta)}{\left(\frac{c_1 g_0}{b_0}\right)\exp(\xi) + g_0 + \left(\frac{g_0 d_1}{b_0}\right)\exp(\eta)}, \tag{16}$$

and

$$v = \frac{\left(\frac{c_1 a_0}{b_0}\right)\exp(\xi) + a_0 + \left(-\frac{a_0 d_1}{b_0}\right)\exp(\eta)}{c_1 \exp(\xi) + b_0 + d_1 \exp(\eta)} e^{i(\sigma+\rho)}. \tag{17}$$

*Case* (1.2):

$$\begin{aligned}
&\Bigg\{ h_1 = \frac{g_0 d_1}{b_0}, b_1 = \frac{a_0 d_1}{b_0}, k_1 = k_1, a_1 = -\frac{c_1 a_0}{b_0}, f_1 = \frac{e_0 d_1}{b_0}, g_1 = \frac{c_1 g_0}{b_0},\\
&s_2 = -\frac{\begin{bmatrix} \chi a_0^2 + \gamma_{12} p_2 q_1 b_0^2 + \gamma_{12} p_1 q_2 b_0^2 + \gamma_{12} p_1 q_1 b_0^2 + \gamma_{12} p_2 q_2 b_0^2 + s_1 b_0^2 + \gamma_{11} p_1^2 b_0^2 \\ + 2\gamma_{11} p_1 p_2 b_0^2 + \gamma_{11} p_2^2 b_0^2 + 2\gamma_{22} q_1 q_2 b_0^2 + \gamma_{22} q_1^2 b_0^2 + \gamma_{22} q_2^2 b_0^2 \end{bmatrix}}{b_0^2},\\
&e_1 = \frac{c_1 (8\beta_2 l_1 a_0^2 g_0 + \alpha_{22} k_1 l_1^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_1 e_0 b_0^2)}{b_0^3 k_1 (\alpha_{22} l_1^2 + \alpha_{11})}, s_1 = s_1, q_1 = q_1,\\
&q_2 = q_2, p_1 = p_1, p_2 = p_2, d_1 = d_1, l_1 = l_1, l_2 = l_2, c_1 = c_1, k_2 = 0 \Bigg\}.
\end{aligned}$$

Substituting these values in Eqs. (7) and (8), we get

$$u = \frac{\left(\frac{c_1(8\beta_2 l_1 a_0^2 g_0 + \alpha_{22} k_1 l_1^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_1 e_0 b_0^2)}{b_0^3 k_1(\alpha_{22} l_1^2 + \alpha_{11})}\right)\exp(\xi) + e_0 + \left(\frac{e_0 d_1}{b_0}\right)\exp(\eta)}{\left(\frac{c_1 g_0}{b_0}\right)\exp(\xi) + g_0 + \left(\frac{g_0 d_1}{b_0}\right)\exp(\eta)}, \tag{18}$$

and

$$v = \frac{\left(-\frac{c_1 a_0}{b_0}\right)\exp(\xi) + a_0 + \left(\frac{a_0 d_1}{b_0}\right)\exp(\eta)}{c_1\exp(\xi) + b_0 + d_1\exp(\eta)} e^{i(\sigma+\rho)}. \tag{19}$$

*Case* (1.3):

$$\Bigg\{ h_1 = \frac{g_0 d_1}{b_0}, b_1 = -\frac{a_0 d_1}{b_0}, k_1 = k_2, a_1 = -\frac{g_1 a_0}{g_0}, c_1 = \frac{g_1 b_0}{g_0},$$

$$s_2 = -\frac{\left[\begin{array}{c}\chi a_0^2 + \gamma_{12} p_2 q_1 b_0^2 + \gamma_{12} p_1 q_2 b_0^2 + \gamma_{12} p_1 q_1 b_0^2 + \gamma_{12} p_2 q_2 b_0^2 + s_1 b_0^2 + \gamma_{11} p_1^2 b_0^2 \\ + 2\gamma_{11} p_1 p_2 b_0^2 + \gamma_{11} p_2^2 b_0^2 + 2\gamma_{22} q_1 q_2 b_0^2 + \gamma_{22} q_1^2 b_0^2 + \gamma_{22} q_2^2 b_0^2\end{array}\right]}{b_0^2},$$

$$f_1 = \frac{d_1(8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^3 k_2(\alpha_{22} l_2^2 + \alpha_{11})}, s_1 = s_1,$$

$$e_1 = \frac{g_1(8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^2 k_2 g_0(\alpha_{22} l_2^2 + \alpha_{11})}, q_1 = q_1,$$

$$q_2 = q_2, p_1 = p_1, p_2 = p_2, d_1 = d_1, l_1 = l_2, l_2 = l_2, g_1 = g_1, k_2 = k_2 \Bigg\}.$$

Combining the above values of parameters with Eqs. (7) and (8), we get the following exact solutions,

$$u = \frac{\left[\begin{array}{c}\left(\frac{g_1(8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^2 k_2 g_0(\alpha_{22} l_2^2 + \alpha_{11})}\right)\exp(\xi) + e_0 \\ + \left(\frac{d_1(8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^3 k_2(\alpha_{22} l_2^2 + \alpha_{11})}\right)\exp(\eta)\end{array}\right]}{\left[g_1\exp(\xi) + g_0 + \left(\frac{g_0 d_1}{b_0}\right)\exp(\eta)\right]}, \tag{20}$$

and

$$v = \frac{\left(-\frac{g_1 a_0}{g_0}\right)\exp(\xi) + a_0 + \left(-\frac{a_0 d_1}{b_0}\right)\exp(\eta)}{\left(\frac{g_1 b_0}{g_0}\right)\exp(\xi) + b_0 + d_1\exp(\eta)} e^{i(\sigma+\rho)}. \tag{21}$$

*Case* (1.4):

$$\Bigg\{ h_1 = \frac{g_0 d_1}{b_0}, b_1 = -\frac{a_0 d_1}{b_0}, k_1 = k_2, a_1 = -\frac{c_1 a_0}{b_0}, g_1 = \frac{c_1 g_0}{b_0},$$

$$s_2 = -\frac{\left[\begin{array}{c}\chi a_0^2 + \gamma_{12} p_2 q_1 b_0^2 + \gamma_{12} p_1 q_2 b_0^2 + \gamma_{12} p_1 q_1 b_0^2 + \gamma_{12} p_2 q_2 b_0^2 + s_1 b_0^2 + \gamma_{11} p_1^2 b_0^2 \\ + 2\gamma_{11} p_1 p_2 b_0^2 + \gamma_{11} p_2^2 b_0^2 + 2\gamma_{22} q_1 q_2 b_0^2 + \gamma_{22} q_1^2 b_0^2 + \gamma_{22} q_2^2 b_0^2\end{array}\right]}{b_0^2},$$

$$f_1 = \frac{d_1(8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^3 k_2(\alpha_{22} l_2^2 + \alpha_{11})}, s_1 = s_1,$$

$$e_1 = \frac{c_1(8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^3 k_2(\alpha_{22} l_2^2 + \alpha_{11})}, q_1 = q_1,$$

$$\left. q_2 = q_2, p_1 = p_1, p_2 = p_2, d_1 = d_1, l_1 = l_2, l_2 = l_2, c_1 = c_1, k_2 = k_2 \right\}.$$

Using the constant parameters in Eqs. (7) and (8) recast into the following as,

$$u = \frac{\left[\begin{array}{c}\left(\frac{c_1(8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^3 k_2(\alpha_{22} l_2^2 + \alpha_{11})}\right)\exp(\xi) + e_0 \\ + \left(\frac{d_1(8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^3 k_2(\alpha_{22} l_2^2 + \alpha_{11})}\right)\exp(\eta)\end{array}\right]}{\left(\frac{c_1 g_0}{b_0}\right)\exp(\xi) + g_0 + \left(\frac{g_0 d_1}{b_0}\right)\exp(\eta)}, \tag{22}$$

and

$$v = \frac{\left(-\frac{c_1 a_0}{b_0}\right)\exp(\xi) + a_0 + \left(-\frac{a_0 d_1}{b_0}\right)\exp(\eta)}{c_1\exp(\xi) + b_0 + d_1\exp(\eta)} e^{i(\sigma+\rho)}. \tag{23}$$

*Case* (1.5):

$$\left\{ d_1 = \frac{b_0 h_1}{g_0}, b_1 = -\frac{a_0 h_1}{g_0}, k_1 = k_2, a_1 = -\frac{c_1 a_0}{b_0}, g_1 = \frac{c_1 g_0}{b_0}, \right.$$

$$s_2 = -\frac{\left[\begin{array}{c}\chi a_0^2 + \gamma_{12} p_2 q_1 b_0^2 + \gamma_{12} p_1 q_2 b_0^2 + \gamma_{12} p_1 q_1 b_0^2 + \gamma_{12} p_2 q_2 b_0^2 + s_1 b_0^2 + \gamma_{11} p_1^2 b_0^2 \\ + 2\gamma_{11} p_1 p_2 b_0^2 + \gamma_{11} p_2^2 b_0^2 + 2\gamma_{22} q_1 q_2 b_0^2 + \gamma_{22} q_1^2 b_0^2 + \gamma_{22} q_2^2 b_0^2\end{array}\right]}{b_0^2},$$

$$f_1 = \frac{h_1(8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^2 k_2 g_0(\alpha_{22} l_2^2 + \alpha_{11})}, s_1 = s_1,$$

$$e_1 = \frac{c_1(8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^3 k_2(\alpha_{22} l_2^2 + \alpha_{11})}, q_1 = q_1,$$

$$\left. q_2 = q_2, p_1 = p_1, p_2 = p_2, h_1 = h_1, l_1 = l_2, l_2 = l_2, c_1 = c_1, k_2 = k_2 \right\}.$$

On substitution of the above values in Eqs. (7) and (8) leads to,

$$u = \frac{\left[\begin{array}{c}\left(\frac{c_1(8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^3 k_2(\alpha_{22} l_2^2 + \alpha_{11})}\right)\exp(\xi) + e_0 \\ + \left(\frac{h_1(8\beta_2 l_2 a_0^2 g_0 + \alpha_{22} k_2 l_2^2 e_0 b_0^2 + 8\beta_1 a_0^2 g_0 + \alpha_{11} k_2 e_0 b_0^2)}{b_0^2 k_2 g_0(\alpha_{22} l_2^2 + \alpha_{11})}\right)\exp(\eta)\end{array}\right]}{\left(\frac{c_1 g_0}{b_0}\right)\exp(\xi) + g_0 + h_1\exp(\eta)}, \tag{24}$$

and

$$v = \frac{\left(-\frac{c_1 a_0}{b_0}\right)\exp(\xi) + a_0 + \left(-\frac{a_0 h_1}{g_0}\right)\exp(\eta)}{c_1\exp(\xi) + b_0 + \left(\frac{b_0 h_1}{g_0}\right)\exp(\eta)} e^{i(\sigma+\rho)}. \tag{25}$$

In order to generate the other class of solutions we assume the following set of parameters and by the same approach as demonstrated above, we solve the equations as follows.

*Case* (2): $a_1 = b_1 = c_1 = d_1 = e_1 = f_1 = g_1 = h_1 = 0$.

In view of the above values, we obtain the equations in exponential powers as,

$$\begin{aligned}
&Q_1 \exp(-2\xi - 4\eta) + Q_2 \exp(-\eta - 4\xi) + Q_3 \exp(-3\xi - 2\eta) + Q_4 \exp(-3\xi - 3\eta)\\
&\quad + Q_5 \exp(-5\xi - \eta) + Q_6 \exp(-5\xi) + Q_7 \exp(-4\xi - 2\eta) + Q_8 \exp(-5\eta)\\
&\quad + Q_9 \exp(-4\eta - \xi) + Q_{10} \exp(-2\xi - 3\eta) + Q_{11} \exp(-\xi - 5\eta) + Q_{12} \exp(-3\xi)\\
&\quad + Q_{13} \exp(-3\eta) + Q_{14} \exp(-2\xi) + Q_{15} \exp(-3\eta - \xi) + Q_{16} \exp(-\xi - \eta)\\
&\quad + Q_{17} \exp(-2\eta) + Q_{18} \exp(-4\eta) + Q_{19} \exp(-\eta - 3\xi) + Q_{20} \exp(-\eta - 2\xi)\\
&\quad + Q_{21} \exp(-\eta) + Q_{22} \exp(-2\xi - 2\eta) + Q_{23} \exp(-\xi) + Q_{24} \exp(-2\eta - \xi) + Q_{25} \exp(-4\xi) = 0,\\
&\text{and}\\
&R_1 \exp(-2\xi - 2\eta) + R_2 \exp(-2\eta - \xi) + R_3 \exp(-3\xi - 2\eta) + R_4 \exp(-4\xi)\\
&\quad + R_5 \exp(-3\xi) + R_6 \exp(-\eta - 4\xi) + R_7 \exp(-5\xi) + R_8 \exp(-2\xi - 3\eta)\\
&\quad + R_9 \exp(-5\eta) + R_{10} \exp(-4\eta) + R_{11} \exp(-4\eta - \xi) + R_{12} \exp(-3\eta - \xi)\\
&\quad + R_{13} \exp(-\xi - \eta) + R_{14} \exp(-2\eta) + R_{15} \exp(-2\xi) + R_{16} \exp(-3\eta)\\
&\quad + R_{17} \exp(-\eta - 2\xi) + R_{18} \exp(-\eta) + R_{19} \exp(-\xi) + R_{20} \exp(-\eta - 3\xi) + R_{21} = 0,
\end{aligned} \tag{26}$$

where $Q_i$ and $R_j$ ($i = 1, \ldots, 25$ & $j = 1, \ldots, 21$) are the determined coefficients. Equating the coefficients of exponentials to zero, we obtain the series of equations as,

$$\begin{aligned}
&Q_1 = 0,\\
&Q_2 = 0,\\
&\vdots\\
&Q_{25} = 0,\\
&\text{and}\\
&R_1 = 0,\\
&R_2 = 0,\\
&\vdots\\
&R_{20} = 0.
\end{aligned} \tag{27}$$

Further solving the above set of equations with the aid of Maple, we can distinguish different cases in the following way,

*Case* (2.1):

$$\begin{aligned}
&\Bigg\{ h_{-1} = \frac{g_0 d_{-1}}{b_0}, b_{-1} = \frac{a_0 d_{-1}}{b_0}, k_1 = k_1, a_{-1} = -\frac{g_{-1} a_0}{b_0}, f_{-1} = \frac{e_0 d_{-1}}{b_0}, g_{-1} = g_{-1},\\
&\quad s_1 = -\frac{\begin{bmatrix} \gamma_{12} p_2 q_1 b_0^2 + s_2 b_0^2 + \gamma_{12} p_1 q_1 b_0^2 + \gamma_{12} p_2 q_2 b_0^2 + \chi a_0^2 + \gamma_{11} p_2^2 b_0^2 + \gamma_{11} p_1^2 b_0^2 \\ + 2\gamma_{11} p_1 p_2 b_0^2 + \gamma_{22} q_2^2 b_0^2 + 2\gamma_{22} q_1 q_2 b_0^2 + \gamma_{22} q_1^2 b_0^2 + \gamma_{12} p_1 q_2 b_0^2 \end{bmatrix}}{b_0^2},\\
&\quad e_{-1} = \frac{g_{-1}(-8\beta_2 l_1 a_0^2 g_0 + \alpha_{22} k_1 l_1^2 e_0 b_0^2 - 8\beta_1 a_0^2 g_0 + \alpha_{11} k_1 e_0 b_0^2)}{b_0^2 k_1 g_0 (\alpha_{22} l_1^2 + \alpha_{11})}, s_2 = s_2, q_1 = q_1,\\
&\quad q_2 = q_2, p_1 = p_1, p_2 = p_2, d_{-1} = d_{-1}, l_1 = l_1, l_2 = l_2, c_{-1} = \frac{b_0 g_{-1}}{g_0}, k_2 = 0 \Bigg\}.
\end{aligned}$$

Using the above constant parameters, we rewrite Eqs. (7) and (8) as

$$u = \frac{\left( \frac{g_{-1}(-8\beta_2 l_1 a_0^2 g_0 + \alpha_{22} k_1 l_1^2 e_0 b_0^2 - 8\beta_1 a_0^2 g_0 + \alpha_{11} k_1 e_0 b_0^2)}{b_0^2 k_1 g_0 (\alpha_{22} l_1^2 + \alpha_{11})} \right) \exp(\xi) + e_0 + \left( \frac{e_0 d_{-1}}{b_0} \right) \exp(\eta)}{g_{-1} \exp(\xi) + g_0 + \left( \frac{g_0 d_{-1}}{b_0} \right) \exp(\eta)}, \tag{28}$$

and

$$v=\frac{\left(-\frac{g_{-1}a_0}{b_0}\right)\exp(\xi)+a_0+\left(\frac{a_0d_{-1}}{b_0}\right)\exp(\eta)}{\left(\frac{b_0g_{-1}}{g_0}\right)\exp(\xi)+b_0+d_{-1}\exp(\eta)}e^{i(\sigma+\rho)}. \tag{29}$$

*Case* (2.2):

$$\Bigg\{h_{-1}=\frac{g_0d_{-1}}{b_0}, b_{-1}=-\frac{a_0d_{-1}}{b_0}, k_1=0, a_{-1}=-\frac{g_{-1}a_0}{g_0}, e_{-1}=\frac{e_0g_{-1}}{g_0}, g_{-1}=g_{-1},$$

$$s_1=-\frac{\begin{bmatrix}\gamma_{12}p_2q_1b_0^2+s_2b_0^2+\gamma_{12}p_1q_1b_0^2+\gamma_{12}p_2q_2b_0^2+\chi a_0^2+\gamma_{11}p_2^2b_0^2+\gamma_{11}p_1^2b_0^2\\ +2\gamma_{11}p_1p_2b_0^2+\gamma_{22}q_2^2b_0^2+2\gamma_{22}q_1q_2b_0^2+\gamma_{22}q_1^2b_0^2+\gamma_{12}p_1q_2b_0^2\end{bmatrix}}{b_0^2},$$

$$f_{-1}=\frac{d_{-1}(-8\beta_2l_2a_0^2g_0+\alpha_{22}k_2l_2^2e_0b_0^2-8\beta_1a_0^2g_0+\alpha_{11}k_2e_0b_0^2)}{b_0^3k_2(\alpha_{22}l_1^2+\alpha_{11})}, s_2=s_2, q_1=q_1,$$

$$q_2=q_2, p_1=p_1, p_2=p_2, d_{-1}=d_{-1}, l_1=l_1, l_2=l_2, c_{-1}=\frac{b_0g_{-1}}{g_0}, k_2=k_2\Bigg\}.$$

Upon substitution of above parameters in Eqs. (7) and (8) yields

$$u=\frac{\left(\frac{e_0g_{-1}}{g_0}\right)\exp(\xi)+e_0+\left(\frac{d_{-1}(-8\beta_2l_2a_0^2g_0+\alpha_{22}k_2l_2^2e_0b_0^2-8\beta_1a_0^2g_0+\alpha_{11}k_2e_0b_0^2)}{b_0^3k_2(\alpha_{22}l_1^2+\alpha_{11})}\right)\exp(\eta)}{g_{-1}\exp(\xi)+g_0+\left(\frac{g_0d_{-1}}{b_0}\right)\exp(\eta)}, \tag{30}$$

and

$$v=\frac{\left(-\frac{g_{-1}a_0}{g_0}\right)\exp(\xi)+a_0+\left(-\frac{a_0d_{-1}}{b_0}\right)\exp(\eta)}{\left(\frac{b_0g_{-1}}{g_0}\right)\exp(\xi)+b_0+d_{-1}\exp(\eta)}e^{i(\sigma+\rho)}. \tag{31}$$

*Case* (2.3):

$$\Bigg\{h_{-1}=\frac{g_0d_{-1}}{b_0}, b_{-1}=-\frac{a_0d_{-1}}{b_0}, k_1=k_2, a_{-1}=-\frac{g_{-1}a_0}{g_0}, g_{-1}=g_{-1},$$

$$s_1=-\frac{\begin{bmatrix}\gamma_{12}p_2q_1b_0^2+s_2b_0^2+\gamma_{12}p_1q_1b_0^2+\gamma_{12}p_2q_2b_0^2+\chi a_0^2+\gamma_{11}p_2^2b_0^2+\gamma_{11}p_1^2b_0^2\\ +2\gamma_{11}p_1p_2b_0^2+\gamma_{22}q_2^2b_0^2+2\gamma_{22}q_1q_2b_0^2+\gamma_{22}q_1^2b_0^2+\gamma_{12}p_1q_2b_0^2\end{bmatrix}}{b_0^2},$$

$$e_{-1}=\frac{g_{-1}(\alpha_{22}k_2l_2^2e_0b_0^2-8\beta_2a_0^2l_2g_0+\alpha_{11}k_2e_0b_0^2-8\beta_1a_0^2g_0)}{b_0^2k_2g_0(\alpha_{22}l_2^2+\alpha_{11})}, q_1=q_1,$$

$$f_{-1}=\frac{d_{-1}(-8\beta_2l_2a_0^2g_0+\alpha_{22}k_2l_2^2e_0b_0^2-8\beta_1a_0^2g_0+\alpha_{11}k_2e_0b_0^2)}{b_0^3k_2(\alpha_{22}l_2^2+\alpha_{11})}, s_2=s_2,$$

$$q_2=q_2, p_1=p_1, p_2=p_2, d_{-1}=d_{-1}, l_1=l_2, l_2=l_2, c_{-1}=\frac{b_0g_{-1}}{g_0}, k_2=k_2\Bigg\}.$$

On substitution of the values, the Eqs. (7) and (8) become

$$u=\frac{\left[\begin{array}{l}\left(\dfrac{g_{-1}(\alpha_{22}k_2l_2^2e_0b_0^2-8\beta_2a_0^2l_2g_0+\alpha_{11}k_2e_0b_0^2-8\beta_1a_0^2g_0)}{b_0^2k_2g_0(\alpha_{22}l_2^2+\alpha_{11})}\right)\exp(\xi)+e_0\\ \quad+\left(\dfrac{d_{-1}(-8\beta_2l_2a_0^2g_0+\alpha_{22}k_2l_2^2e_0b_0^2-8\beta_1a_0^2g_0+\alpha_{11}k_2e_0b_0^2)}{b_0^3k_2(\alpha_{22}l_2^2+\alpha_{11})}\right)\exp(\eta)\end{array}\right]}{g_{-1}\exp(\xi)+g_0+\left(\frac{g_0d_{-1}}{b_0}\right)\exp(\eta)},\tag{32}$$

and

$$v=\frac{\left(-\frac{g_{-1}a_0}{g_0}\right)\exp(\xi)+a_0+\left(-\frac{a_0d_{-1}}{b_0}\right)\exp(\eta)}{\left(\frac{b_0g_{-1}}{g_0}\right)\exp(\xi)+b_0+d_{-1}\exp(\eta)}e^{i(\sigma+\rho)}.\tag{33}$$

*Case* (2.4):

$$\left\{h_{-1}=\frac{g_0d_{-1}}{b_0},b_{-1}=-\frac{a_0d_{-1}}{b_0},k_1=0,a_{-1}=-\frac{c_{-1}a_0}{b_0},e_{-1}=\frac{e_0c_{-1}}{b_0},g_{-1}=\frac{c_{-1}g_0}{b_0},\right.$$

$$s_2=-\frac{\left[\begin{array}{c}\gamma_{12}p_2q_1b_0^2+s_1b_0^2+\gamma_{12}p_1q_1b_0^2+\gamma_{12}p_2q_2b_0^2+\chi a_0^2+\gamma_{11}p_2^2b_0^2+\gamma_{11}p_1^2b_0^2\\+2\gamma_{11}p_1p_2b_0^2+\gamma_{22}q_2^2b_0^2+2\gamma_{22}q_1q_2b_0^2+\gamma_{22}q_1^2b_0^2+\gamma_{12}p_1q_2b_0^2\end{array}\right]}{b_0^2},$$

$$f_{-1}=\frac{d_{-1}(-8\beta_2l_2a_0^2g_0+\alpha_{22}k_2l_2^2e_0b_0^2-8\beta_1a_0^2g_0+\alpha_{11}k_2e_0b_0^2)}{b_0^3k_2(\alpha_{22}l_2^2+\alpha_{11})},s_1=s_1,q_1=q_1,$$

$$\left.q_2=q_2,p_1=p_1,p_2=p_2,d_{-1}=d_{-1},l_1=l_1,l_2=l_2,c_{-1}=c_{-1},k_2=k_2\right\}.$$

We rewrite Eqs. (7) and (8) after substituting above values as

$$u=\frac{\left(\frac{e_0g_{-1}}{g_0}\right)\exp(\xi)+e_0+\left(\frac{d_{-1}(-8\beta_2l_2a_0^2g_0+\alpha_{22}k_2l_2^2e_0b_0^2-8\beta_1a_0^2g_0+\alpha_{11}k_2e_0b_0^2)}{b_0^3k_2(\alpha_{22}l_1^2+\alpha_{11})}\right)\exp(\eta)}{g_{-1}\exp(\xi)+g_0+\left(\frac{g_0d_{-1}}{b_0}\right)\exp(\eta)},\tag{34}$$

and

$$v=\frac{\left(-\frac{g_{-1}a_0}{g_0}\right)\exp(\xi)+a_0+\left(-\frac{a_0d_{-1}}{b_0}\right)\exp(\eta)}{\left(\frac{b_0g_{-1}}{g_0}\right)\exp(\xi)+b_0+d_{-1}\exp(\eta)}e^{i(\sigma+\rho)}.\tag{35}$$

*Case* (2.5):

$$\left\{h_{-1}=\frac{g_0d_{-1}}{b_0},b_{-1}=-\frac{a_0d_{-1}}{b_0},k_1=0,a_{-1}=-\frac{c_{-1}a_0}{b_0},e_{-1}=\frac{e_0c_{-1}}{b_0},g_{-1}=\frac{c_{-1}g_0}{b_0},\right.$$

$$s_1=-\frac{\left[\begin{array}{c}\gamma_{12}p_2q_1b_0^2+s_2b_0^2+\gamma_{12}p_1q_1b_0^2+\gamma_{12}p_2q_2b_0^2+\chi a_0^2+\gamma_{11}p_2^2b_0^2+\gamma_{11}p_1^2b_0^2\\+2\gamma_{11}p_1p_2b_0^2+\gamma_{22}q_2^2b_0^2+2\gamma_{22}q_1q_2b_0^2+\gamma_{22}q_1^2b_0^2+\gamma_{12}p_1q_2b_0^2\end{array}\right]}{b_0^2},$$

$$f_{-1}=\frac{d_{-1}(-8\beta_2l_2a_0^2g_0+\alpha_{22}k_2l_2^2e_0b_0^2-8\beta_1a_0^2g_0+\alpha_{11}k_2e_0b_0^2)}{b_0^3k_2(\alpha_{22}l_2^2+\alpha_{11})},s_2=s_2,q_1=q_1,$$

$$\left.q_2=q_2,p_1=p_1,p_2=p_2,d_{-1}=d_{-1},l_1=l_1,l_2=l_2,c_{-1}=c_{-1},k_2=k_2\right\}.$$

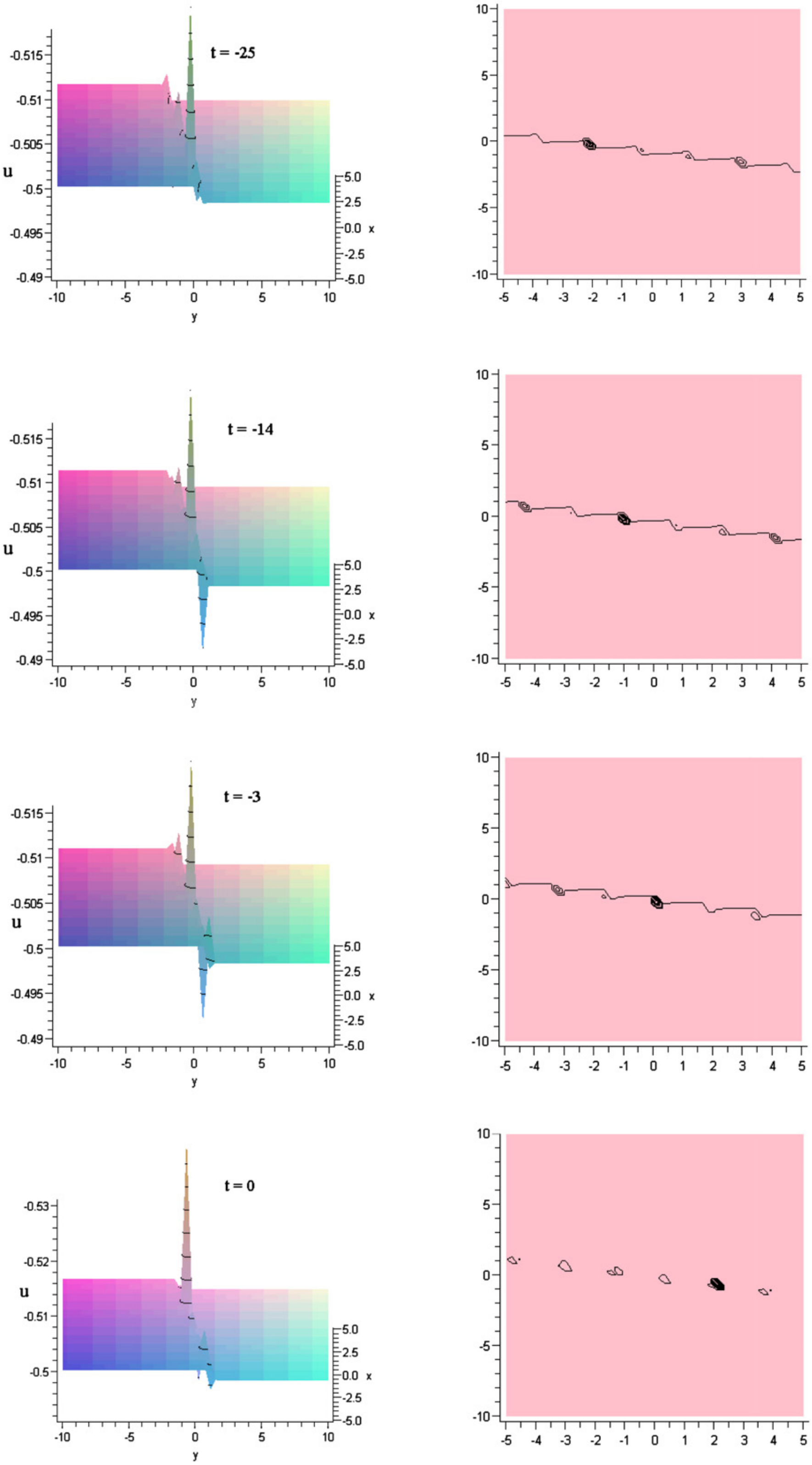


**Fig. 1.** Propagating multidromion-like solutions $u(x, y, t)$ for various instants of time.

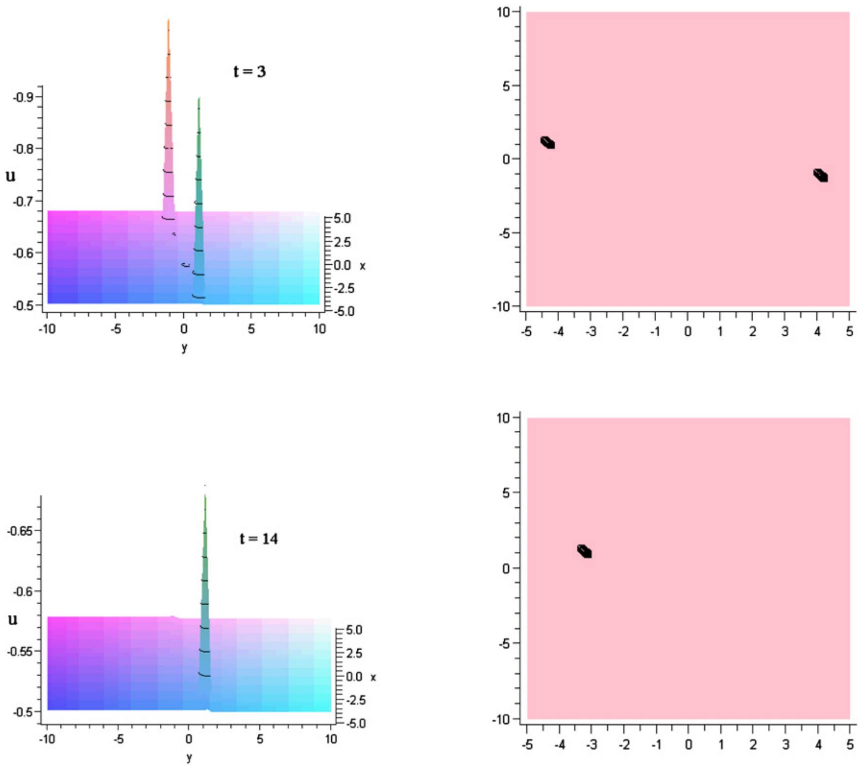


**Fig. 1.** (*continued*)

Using the values in Eqs. (7) and (8) the solution reads

$$u=\frac{\left(\frac{e_0c_{-1}}{b_0}\right)\exp(\xi)+e_0+\left(\frac{d_{-1}(-8\beta_2l_2a_0^2g_0+\alpha_{22}k_2l_2^2e_0b_0^2-8\beta_1a_0^2g_0+\alpha_{11}k_2e_0b_0^2)}{b_0^3k_2(\alpha_{22}l_2^2+\alpha_{11})}\right)\exp(\eta)}{\left(\frac{c_{-1}g_0}{b_0}\right)\exp(\xi)+g_0+\left(\frac{g_0d_{-1}}{b_0}\right)\exp(\eta)},\tag{36}$$

and

$$v=\frac{\left(-\frac{c_{-1}a_0}{b_0}\right)\exp(\xi)+a_0+\left(-\frac{a_0d_{-1}}{b_0}\right)\exp(\eta)}{c_{-1}\exp(\xi)+b_0+d_{-1}\exp(\eta)}e^{i(\sigma+\rho)}.\tag{37}$$

*Case* (2.6):

$$\Bigg\{h_{-1}=\frac{g_0d_{-1}}{b_0},b_{-1}=\frac{a_0d_{-1}}{b_0},k_1=k_1,a_{-1}=-\frac{c_{-1}a_0}{b_0},f_{-1}=\frac{e_0d_{-1}}{b_0},g_{-1}=\frac{c_{-1}g_0}{b_0},$$

$$s_1=-\frac{\begin{bmatrix}\gamma_{12}p_2q_1b_0^2+s_2b_0^2+\gamma_{12}p_1q_1b_0^2+\gamma_{12}p_2q_2b_0^2+\chi a_0^2+\gamma_{11}p_2^2b_0^2+\gamma_{11}p_1^2b_0^2\\ +2\gamma_{11}p_1p_2b_0^2+\gamma_{22}q_2^2b_0^2+2\gamma_{22}q_1q_2b_0^2+\gamma_{22}q_1^2b_0^2+\gamma_{12}p_1q_2b_0^2\end{bmatrix}}{b_0^2},$$

$$e_{-1}=\frac{c_{-1}(-8\beta_2l_2a_0^2g_0+\alpha_{22}k_2l_2^2e_0b_0^2-8\beta_1a_0^2g_0+\alpha_{11}k_2e_0b_0^2)}{b_0^3k_2(\alpha_{22}l_2^2+\alpha_{11})},s_2=s_2,q_1=q_1,$$

$$q_2=q_2,p_1=p_1,p_2=p_2,d_{-1}=d_{-1},l_1=l_1,l_2=l_2,c_{-1}=c_{-1},k_2=0\Bigg\}.$$

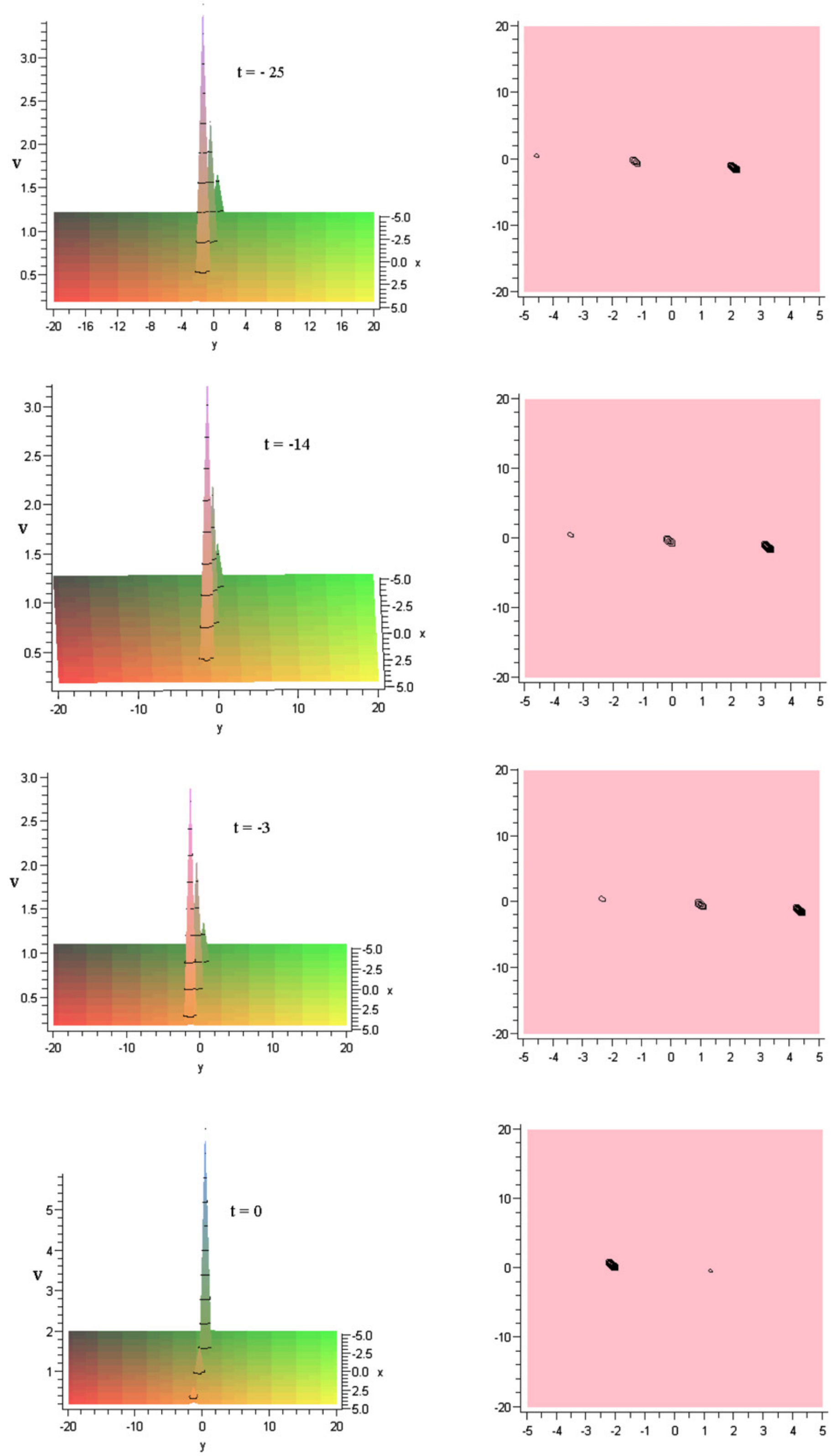


**Fig. 2.** Propagating multidromion-like solutions $v(x, y, t)$ for various instants of time.

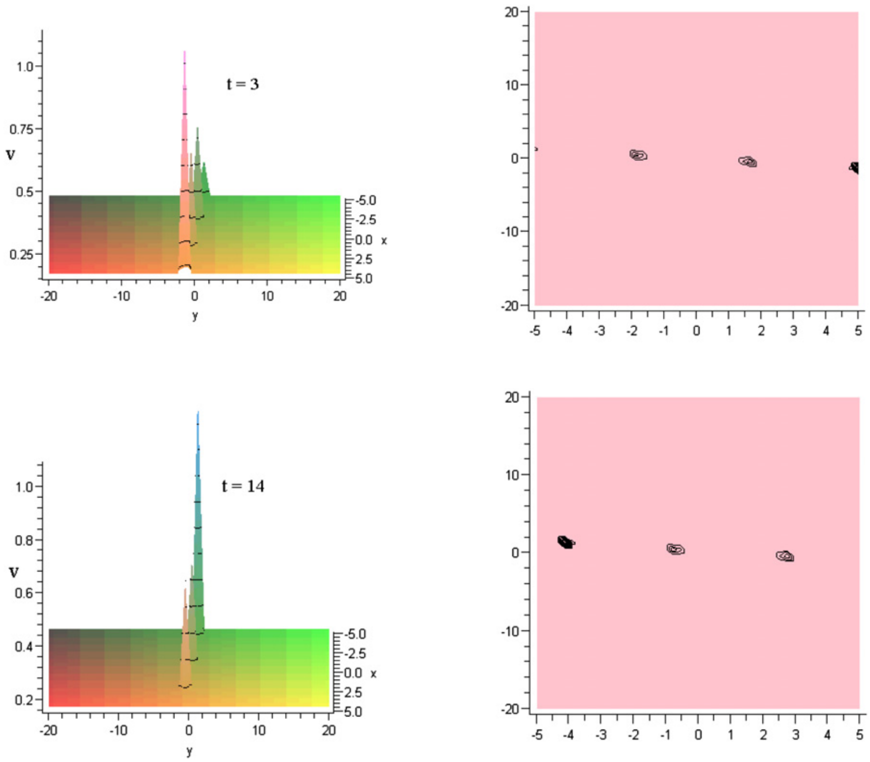


**Fig. 2.** (*continued*)

Upon substituting the values in Eqs. (7) and (8), the form of exact solutions reads

$$u=\frac{\left(\frac{c_{-1}(-8\beta_2 l_2 a_0^2 g_0+\alpha_{22}k_2 l_2^2 e_0 b_0^2-8\beta_1 a_0^2 g_0+\alpha_{11}k_2 e_0 b_0^2)}{b_0^3 k_2(\alpha_{22}l_2^2+\alpha_{11})}\right)\exp(\xi)+e_0+\left(\frac{e_0 d_{-1}}{b_0}\right)\exp(\eta)}{\left(\frac{c_{-1}g_0}{b_0}\right)\exp(\xi)+g_0+\left(\frac{g_0 d_{-1}}{b_0}\right)\exp(\eta)}, \tag{38}$$

and

$$v=\frac{\left(-\frac{c_{-1}a_0}{b_0}\right)\exp(\xi)+a_0+\left(\frac{a_0 d_{-1}}{b_0}\right)\exp(\eta)}{c_{-1}\exp(\xi)+b_0+d_{-1}\exp(\eta)}e^{i(\sigma+\rho)}. \tag{39}$$

The schematic form of the $(2+1)$-dimensional dromion-like exponentially localized structures as represented in case (1.3) are depicted along with its contour plots in Figs. 1–3 for various values of time and for the choices of parameters $q_1'=1.0467$, $q_2'=1.57$, $j_2=0.5$, $j_3=0.1$, $j_4=0.1$, $l_2=3.8$, $k_2=6.8$, $l_1=l_2$, $k_1=k_2$, $\omega=1.75$, $p_1=0.75$, $p_2=0.5$, $q_1=0.5$, $q_2=0.25$, $s_1=0.75$, $a_0=0.1$, $b_0=0.6$, $e_0=0.1$, $g_0=-0.2$, $d_1=5$ and $g_1=5$. From the figures it is evident that both the solutions for $u(x,y,t)$ and $v(x,y,t)$ are in general exhibiting steadily propagating patterns without showing any appreciable change in phase and moving only along $x$ direction with finite amplitude and velocity. In Fig. 1, at time $t=-14$, the solution $u$ which represents the mean motion induced by the oscillatory wave packet demonstrates a mixed phase of dromion-like structure with oscillatory kink tails which looks similar to the mean flow mentioned by Chow and Lou, [36] for a DS system in the hydrodynamic context. The contour plots for $u$ at instants of time $t=-25,-14,-3$ and 0 clearly portray the hybrid phase of kink-dromion-like localized structure resembling the kink excitations of the sine-Gordon model. Further, from Fig. 1, one can understand that as time progresses the central part of the kink slowly exhibits an anti-steepening motion when $t=3$, the phase difference between the tails vanish and thus relieving the solution from the hybrid phase as clearly seen in the corresponding contour. A close inspection of the plots reveals one peculiar phenomenon that the solutions reach the maximum amplitude just before it gets radiated away. In Fig. 2, the propagating multidromion-like structures radiate energy and eventually dies up. We also have plotted $|v|^2$ of the envelope function as a function of $x$ and $y$ for instants of time $t=-5.9$, 0.0, 1.6 and 5.9 as shown in the Fig. 3. From the figures, it is possible to trace the trend followed by the multidromion-like solutions. Eventually, the profile of solutions experience some fluctuations in the amplitude and more specially at $t=5.9$, the amplitude shoots up high to the maximum followed in principle by the

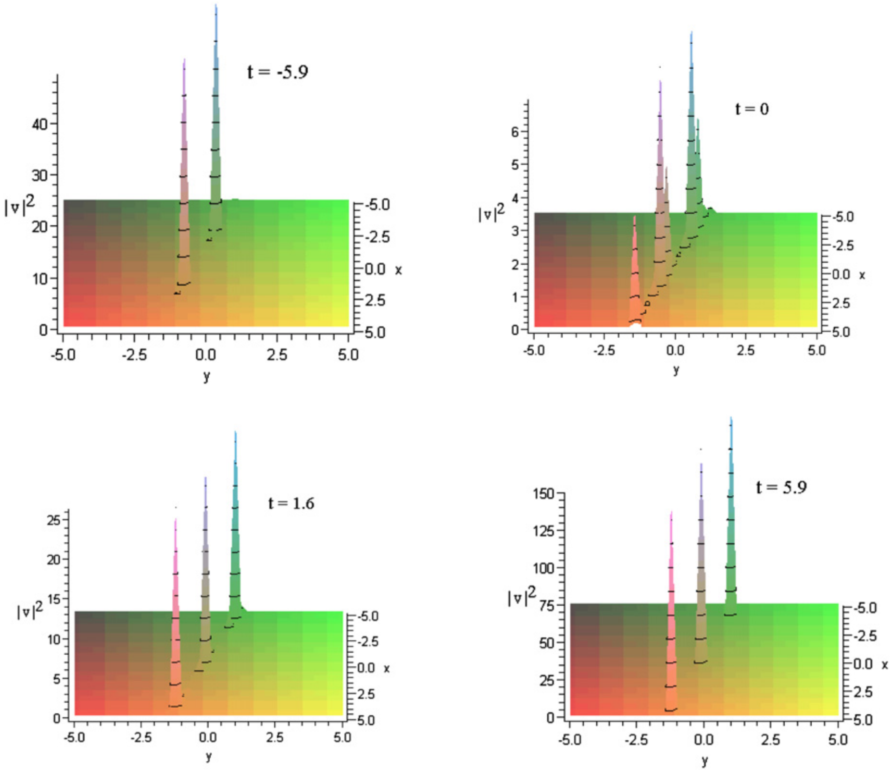


**Fig. 3.** Snap-shots of the envelop of the multidromion-like profile at various instants of time.

annihilation of envelope functions. In these plots, the decaying tracks for the localized structures can be clearly seen. In the same way, we checked the nature of other solutions for various choices of parameters and the same discussion holds except they experience some drastic fluctuations in amplitude.

## 4. Conclusions

In summary, we have constructed exponentially decaying localized multidromion-like structures for the $(2+1)$-dimensional generalized Davey–Stewartson equations, other than integrability conditions governing the dynamics of a weakly nonlinear modulation of wave packets in the higher dimensional lattice with higher order cubic and quartic potentials. The localized solutions suffer both spatially and temporally varying amplitude describing a set of steadily propagating structures at arbitrary velocities and curves at various instants of time before they radiate. We achieved this by invoking the double-Exp method aided with symbolic computation which remains an indispensable tool to deal with computational algebraic systems. It is also anticipated that this investigation on $(2+1)$-dimensional GDS equations may have wider ramifications in hydrodynamics, and a good understanding of the varieties of solutions of GDS may be very helpful for civil engineers to apply a nonlinear water wave model in a harbor and coastal design and also to deal with nonlinear wave coupled with dispersive long gravity waves in two horizontal directions of shallow water of uniform depth.

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